# Force-Balance and Mohr's Equations 

(Mohr Stress)

## The Force-Balance Analysis

In high school physics we were taught that pressure is defined as force per unit area. Stress may have the same definition. The distinction between pressure and stress in geology is based on the nature of the material on which the force is acting. The distinction is made depending on whether the material in question has a shear strength. Materials such as rock are said to have a shear strength because they maintain their shape when placed unsupported on a table. If we are dealing with a rock that has a shear strength (fluids such as water and gases do not), then we say that it exerts a stress on its surroundings. Materials which have a shear strength can exert different stresses in different directions. In contrast, water, without a shear strength would proceed to run over the table top seeking the lowest spot. If we wish to describe the force per unit area that a liquid or gas is exerting on its container, we use the term pressure. Water in the pores in rocks exerts a pressure on the grains surrounding the pore.

The most useful equations for teaching the concept of stress are the equations for normal $\left(\sigma_{\mathrm{n}}\right)$ and shear $(\tau)$ stress in terms of principal stresses $\left(\sigma_{1}, \sigma_{3}\right)$. The derivation of these equations is based on a force-balance problem which assumes that a body subject to forces is in equilibrium which means that all forces in any direction add to zero.

$\sin \theta$
(Fig. 1)
Consider the forces acting parallel to the front face of the triangular solid shown in Figure 1. These forces are acting in a plane so that this force-balance problem is two dimensional. If the area of the hypotenuse face is unity (i.e. 1), then the area of the left face is unity times $\sin \theta$ whereas the area of the bottom face is unity times $\cos \theta$. Forces acting on the three faces are stresses multiplied by the area of the face [Force $=$ (force/area) $\times($ area $)$ ]. Summing the forces in the horizontal and vertical directions (the forces add to zero in both directions), we obtain

$$
\begin{align*}
& \Sigma \mathbf{F}_{\mathrm{h}}=0=\mathbf{F}_{\mathrm{X}}-\mathbf{S} \sin \theta-\mathbf{N} \cos \theta  \tag{1a}\\
& \Sigma \mathbf{F}_{\mathrm{V}}=0=\mathbf{F}_{\mathrm{y}}-\mathbf{N} \sin \theta+\mathbf{S} \cos \theta \tag{1b}
\end{align*}
$$

Rewriting the force terms using components of stress, we obtain the normal and shear stresses acting on a plane an angle of $\theta$ to $\sigma_{1}$ in terms of the principal stresses.

$$
\begin{equation*}
\sigma_{1} \cos \theta-\sigma_{n} \cos \theta-\tau \sin \theta=0 \tag{2a}
\end{equation*}
$$

$$
\begin{equation*}
\sigma_{3} \sin \theta-\sigma_{n} \sin \theta+\tau \cos \theta=0 \tag{2b}
\end{equation*}
$$

In many geological applications the plane of interest is a fault plane. Now we can solve these two equations for $\sigma_{\mathrm{n}}$ and $\tau$

$$
\begin{align*}
& \sigma_{\mathrm{n}}=\sigma_{3} \cos ^{2} \theta+\sigma_{1} \sin ^{2} \theta  \tag{3a}\\
& \tau=\left(\sigma_{1}-\sigma_{3}\right) \cos \theta \sin \theta \tag{3b}
\end{align*}
$$

Remember that

$$
\begin{equation*}
\sin 2 \theta=2 \sin \theta \cos \theta \tag{4}
\end{equation*}
$$

and also

$$
\begin{equation*}
\cos ^{2} \theta=1 / 2(1+\cos 2 \theta) \text { and } \sin ^{2} \theta=1 / 2(1-\cos 2 \theta) \tag{5}
\end{equation*}
$$

We derive

$$
\begin{equation*}
\sigma_{n}=1 / 2\left(\sigma_{1}+\sigma 3\right)+1 / 2\left(\sigma_{1}-\sigma 3\right) \cos 2 \theta \tag{6a}
\end{equation*}
$$

and

$$
\begin{equation*}
\tau=1 / 2\left(\sigma_{1}-\sigma_{3}\right) \sin 2 \theta . \tag{6b}
\end{equation*}
$$

Note that in these equations $\theta$ is the angle between $\sigma 1$ and the normal to the fault plane. This convention leads to a positive sign in the middle of equation 6 . The sign convention is that $\theta$ is the angle between the normal to the fault plane and $\sigma_{1}$.

The general force-balance problem is shown in figure 2 where the back and side faces of the unit triangle are subject to both normal and shear stresses. In other words the coordinate system x-y is not parallel to the principal stress directions. The general case is given to illustrate a property of stress called invariance with respect to coordinate system. In two dimensions we define stress as a force per unit of line length, in contrast to the three dimensional situation where stress is a force per unit area (Fig. 1). Consider a coordinate system $\mathrm{O}_{\mathrm{xy}}$ with an arbitrary line AB cutting the x and y axes such that the normal to the line AB makes an angle $\theta$ with the x -axis. This gives a right triangle AOB with sides OA (parallel to $\mathrm{O}_{\mathrm{x}}$ ) and OB (parallel to $\mathrm{O}_{\mathrm{y}}$ ) and a hypotenuse AB . Across the line AB a stress vector $\mathbf{p}$ can be applied making an angle $\theta$ with the x -axis. Remember that $\mathbf{p}=\delta \mathbf{f} / \delta \mathbf{A}$ when $\delta \mathrm{A} \Rightarrow 0$, so a stress vector can represent stress at a point. Otherwise stress is defined on a line (2-D) or a surface (3-D). The stress vector $\mathbf{p}$ can be resolved into components parallel to the x and y axes: $\mathbf{p}=\mathbf{p}_{\mathrm{x}}+\mathbf{p}_{\mathrm{y}}$. Even though P is called a vector it still has units of stress (force/length in two dimensions). Because the triangle ABO is in equilibrium the sum of the force-vectors on all sides must balance. In 2-D stress multiplied by line length will give a force vector. So

$$
\begin{equation*}
\mathbf{p}_{\mathrm{x}} \mathrm{AB}=\sigma_{\mathrm{x}} \mathrm{OB}+\tau_{\mathrm{yx}} \mathrm{OA} \tag{8}
\end{equation*}
$$


(Fig. 2)

If length $\mathrm{a}=\mathrm{AB}$, then $\mathrm{a} \times \cos \theta=\mathrm{OB}$ and $\mathrm{a} \times \sin \theta=\mathrm{OA}$. If we divide by length a , then

$$
\begin{equation*}
\mathbf{p}_{\mathrm{x}}=\sigma_{\mathrm{x}} \cos \theta+\tau_{\mathrm{yx}} \sin \theta \tag{9a}
\end{equation*}
$$

Likewise

$$
\begin{equation*}
\mathbf{p}_{\mathrm{y}}=\sigma_{\mathrm{y}} \sin \theta+\tau_{\mathrm{xy}} \cos \theta \tag{9b}
\end{equation*}
$$

Now consider a normal stress $\sigma_{\mathrm{n}}$ and shear stress $\tau$ across AB in terms of the components of the stress vector

$$
\begin{equation*}
\sigma_{\mathrm{n}}=\mathbf{p}_{\mathrm{x}} \cos \theta+\mathbf{p}_{\mathrm{y}} \sin \theta \tag{9c}
\end{equation*}
$$

The equality holds for $\tau$

$$
\begin{equation*}
\tau=\mathbf{p}_{\mathrm{y}} \cos \theta-\mathbf{p}_{\mathrm{x}} \sin \theta \tag{9d}
\end{equation*}
$$

Substituting for $\mathbf{p}_{\mathrm{x}}$ and $\mathbf{p}_{\mathrm{y}}$ and remembering that $\sin 2 \theta=2 \sin \theta \cos \theta$ and also $\cos 2 \theta=\cos ^{2} \theta-\sin ^{2} \theta$

$$
\begin{equation*}
\sigma_{\mathbf{n}}=\sigma_{x} \cos ^{2} \theta+2 \tau_{x y} \sin \theta \cos \theta+\sigma_{y} \sin ^{2} \theta \tag{10a}
\end{equation*}
$$

and

$$
\begin{equation*}
\tau=1 / 2\left(\sigma_{\mathbf{y}}-\sigma_{\mathbf{x}}\right) \sin 2 \theta+\tau_{\mathbf{x y}} \cos 2 \theta \tag{10b}
\end{equation*}
$$

These are the general equations for a stress system in which the orientation of the principal stresses are unknown. The following exercise illustrates the case for the principal stresses being parallel to the coordinate axes.

(Fig. 3)
Now we consider the effect of rotating the coordinate system on the values of stress (Fig. 3). To accomplish this coordinate rotation let $\sigma_{\mathrm{n}}$ and $\tau$ be $\sigma_{x^{\prime}}$ and $\tau_{x^{\prime} y^{\prime}}$ where $\mathrm{O}_{\mathrm{x}^{\prime} y^{\prime}}$ is rotated $\theta$ from $\mathrm{O}_{\mathrm{xy}}$. From a previous equation we have

$$
\begin{equation*}
\sigma_{x^{\prime}}=\sigma_{\mathrm{x}} \cos ^{2} \theta+2 \tau_{\mathrm{xy}} \sin \theta \cos \theta+\sigma_{\mathrm{y}} \sin ^{2} \theta \tag{11a}
\end{equation*}
$$

Now we must find $\sigma_{y^{\prime}}$ in the new coordinate system. This is accomplished by replacing $\theta$ by $\theta+1 / 2 \pi$ and we get

$$
\begin{equation*}
\sigma_{y^{\prime}}=\sigma_{\mathrm{x}} \sin ^{2} \theta-2 \tau_{\mathrm{xy}} \sin \theta \cos \theta+\sigma_{\mathrm{y}} \cos ^{2} \theta \tag{11b}
\end{equation*}
$$

If we add $\sigma_{x^{\prime}}$ and $\sigma_{y^{\prime}}$ and remember that $\sin ^{2} \theta+\cos ^{2} \theta=1$, we get

$$
\begin{equation*}
\sigma_{x^{\prime}}+\sigma_{y^{\prime}}=\sigma_{\mathrm{x}}+\sigma_{\mathrm{y}} . \tag{12}
\end{equation*}
$$

This shows that the sum of the normal stresses is invariant or unchanged by rotation of the coordinate system. Likewise

$$
\begin{equation*}
\tau_{x^{\prime} y^{\prime}}=0.5\left(\sigma_{\mathrm{y}}-\sigma_{\mathrm{x}}\right) \sin 2 \theta+\tau_{\mathrm{xy}} \cos 2 \theta \tag{13}
\end{equation*}
$$

Principal stresses are found in planes containing no shear stress. If in these planes, $\tau_{x " y}{ }^{\prime \prime}=0$, then from the previous equation

$$
\begin{equation*}
\tan 2 \Theta=\frac{2 \tau_{\mathrm{xy}}}{\overline{\sigma_{\mathrm{x}}-\sigma_{\mathrm{y}}}} \tag{14}
\end{equation*}
$$

where $\Theta$ is the one angle between the coordinate system $\mathrm{O}_{\mathrm{x}} \mathrm{y}$ " and $\mathrm{O}_{\mathrm{xy}}$ where the shear stresses vanish along the directions $\mathrm{O}_{\mathrm{x}}$ " and $\mathrm{O}_{\mathrm{y}}$ ". In this coordinate system the only stresses are the normal stresses $\sigma_{\mathrm{x}}$ " and $\sigma_{\mathrm{y}}{ }^{\prime \prime}$. This coordinate system contains the principal stress axes and the components, $\sigma_{x}$ " and $\sigma_{y}$ "are known as the principal stresses.

(Fig. 4)
Let us now call these principal stresses $\sigma 1$ and $\sigma_{2}$ (Fig. 4) and then we shall choose the coordinate system such that the x and y axes are in the direction of the principal stresses, $\sigma_{1}$ and $\sigma_{2}$. Now the normal $\sigma_{\mathrm{n}}$ and shear $\tau$ stresses across a line whose normal is inclined at $\theta$ to $\sigma_{1}$ is

$$
\begin{equation*}
\sigma_{\mathrm{n}}=\sigma_{1} \cdot \cos ^{2} \theta+\sigma_{2} \cdot \sin ^{2} \theta \tag{15}
\end{equation*}
$$

Again remember that

$$
\cos ^{2} \theta=1 / 2(1+\cos 2 \theta) \text { and } \sin ^{2} \theta=1 / 2(1-\cos 2 \theta)
$$

we can rearrange the equation for the normal stress

$$
\left.\sigma_{\mathrm{n}}=1 / 2\left(\sigma_{1}+\sigma_{2}\right)+1 / 2\left(\sigma_{1}-\sigma_{2}\right) \cos 2 \theta \quad \quad \text { (repeat of } 6 \mathrm{a}\right)
$$

and shear stress

$$
\begin{equation*}
\tau=1 / 2\left(\sigma_{1}-\sigma_{2}\right) \sin 2 \theta \tag{repeatof6b}
\end{equation*}
$$

From this last equation we see that shear stress is greatest when $\theta=\pi / 4$ and $3 \pi / 4$.
Using this same coordinate system where the axes are parallel to the principal stresses $\sigma_{1}$ and $\sigma_{2}$, we can look at the stress vectors $\mathbf{p}_{\mathrm{x}}$ and $\mathbf{p}_{\mathrm{y}}$. They become

$$
\begin{equation*}
\mathbf{p}_{\mathrm{x}}=\sigma_{1} \cdot \cos \theta \quad \text { and } \quad \mathbf{p}_{\mathrm{y}}=\sigma_{2} \cdot \sin \theta \tag{16}
\end{equation*}
$$


(Fig. 5)
Substituting into the equation $\sin ^{2} \theta+\cos ^{2} \theta=1$, we can generate the equation for an ellipse where the $\mathbf{p}_{\mathrm{x}}$ and $\mathbf{p}_{\mathrm{y}}$ are on the ellipse called the stress ellipse.

$$
\begin{equation*}
\frac{\mathbf{p}_{\mathrm{x}}^{2}}{\sigma_{1}^{2}}+\frac{\mathbf{p y}^{2}}{\sigma_{2}^{2}}=1 \tag{17}
\end{equation*}
$$

The semi axes of the ellipse are $\sigma_{1}$ and $\sigma_{2}$.
The Mohr's circle is a graphical method of representing the state of stress of a rock in two dimensions (Fig. 5). The equations used for the Mohr's circle representation are those derived above for the coordinate system with axes parallel to the principal stresses. The Mohr's circle may be used to derive the normal $\sigma_{\mathrm{n}}$ and shear $\tau$ stresses on any plane whose normal is oriented at $\theta$ from $\sigma_{1}$. The coordinate system for the Mohr's circle representation is $\sigma_{n}$ along the horizontal axes with increasing compression to the right and $\tau$ along the vertical axes. Critical points along the $\sigma_{n}$-axes are $\mathrm{OP}=\sigma_{1}, \mathrm{OQ}=\sigma_{2}$, and $\mathrm{C}=1 / 2\left(\sigma_{1}+\sigma_{2}\right)$. The angle measured as PCA counter clockwise from OP is $2 \theta$. Now we have

$$
\begin{equation*}
\sigma_{\mathrm{n}}=\mathrm{OB}=\mathrm{OC}+\mathrm{CB}=1 / 2\left(\sigma_{1}+\sigma_{2}\right)+1 / 2\left(\sigma_{1}-\sigma_{2}\right) \cos 2 \theta \tag{18a}
\end{equation*}
$$

and

$$
\begin{equation*}
\tau=\mathrm{AB}=1 / 2\left(\sigma_{1}-\sigma_{2}\right) \sin 2 \theta . \tag{18b}
\end{equation*}
$$

